On tridiagonal matrices unitarily equivalent to normal matrices

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Abstract

In this article the unitary equivalence transformation of normal matrices to tridiagonal form is studied.

It is well-known that any matrix is unitarily equivalent to a tridiagonal matrix. In case of a normal matrix the resulting tridiagonal inherits a strong relation between its super- and subdiagonal elements. The corresponding elements of the super- and subdiagonal will have the same absolute value.

In this article some basic facts about a unitary equivalence transformation of an arbitrary matrix to tridiagonal form are firstly studied. Both an iterative reduction based on Krylov sequences as a direct tridiagonalization procedure via Householder transformations are reconsidered. This equivalence transformation is then applied to the normal case and equality of the absolute value between the super- and subdiagonals is proved. Self-adjointness of the resulting tridiagonal matrix with regard to a specific scalar product is proved. Properties when applying the reduction on symmetric, skew-symmetric, Hermitian, skew-Hermitian and unitary matrices and their relations with, e.g., complex symmetric and pseudo-symmetric matrices are presented.

It is shown that the reduction can then be used to compute the singular value decomposition of normal matrices making use of the Takagi factorization. Finally some extra properties of the reduction as well as an efficient method for computing a unitary complex symmetric decomposition of a normal matrix are given.

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It is shown that the reduction can then be used to compute the singular value decomposition of normal matrices making use of the Takagi factorization. Finally some extra properties of the reduction as well as an efficient method for computing a unitary complex symmetric decomposition of a normal matrix are given.

Keywords: normal matrices, complex symmetric matrices, Takagi factorization, tridiagonal matrices, singular values, unitary equivalence, unitary-complex symmetric factorization, Krylov subspaces

1. Introduction

Normal matrices are an important class of matrices arising in various applications and satisfying the following simple commutative relation $AA^H = A^H A$. Hermitian, skew-Hermitian and unitary matrices are all well-known subclasses of the class of normal matrices. Many interesting properties are known about normal matrices [1–5] related to e.g. the eigenvalue and singular value decomposition, the polar decomposition, the Hermitian $H = 1/2(A + A^H)$ and skew-Hermitian part $K = 1/2(A - A^H)$ and their relation with the Toeplitz decomposition $A = H + K$. Also nowadays attention is paid to the class of co-normal matrices [6, 7].

Concerning eigenvalue and singular value methods, many algorithms for the classes of, e.g., Hermitian, skew-Hermitian and unitary matrices are known (see e.g. [8–13]). All these methods consist of two phases. An initial reduction to simpler form $O(n^3)$ is followed by for instance the widespread QR-method for computing all eigenvalues (on average this takes $O(n^3)$ operations[1]).

The most widespread and well-known method for computing singular values is the Golub-Kahan method [16]. Again two steps are required, the so-called Golub-Kahan bidiagonalization procedure followed by a QR-like method.

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[1]For detailed complexity counts we refer to the books [12, 14, 15].

The article [16] describes both a direct method based on Householder reflectors [13], as well as an iterative Lanczos-like method for reducing a matrix to bidiagonal form.

For computing eigenvalues of a Hermitian matrix first a unitary similarity transform is used to tridiagonalize the matrix. The eigenvalues of the resulting tridiagonal matrix can then be computed by either QR-methods, divide and conquer methods, ... [12, 14, 17] (an overview can be found in [18]).

Also for the generic normal case eigenvalue problems have been studied. Iterative methods for computing eigenvalues as well as methods for transforming normal matrices to matrices with growing bandwidth[2] have been proposed in [4, 19–23]. The method proposed in [4, 19] transforms the normal matrix to a band form with increasing bandwidth. In case of Hermitian and skew-Hermitian matrices this approach coincides with the standard tridiagonalization procedures. Unfortunately even though attractive, this approach is not capable of achieving the same complexity as the well-known methods for computing eigenvalues of Hermitian matrices. Computing singular values of normal matrices has not been studied intensively, since the standard Golub-Kahan algorithm is capable of computing all singular values and singular vectors of normal matrices.

Two matrices A and B of the same dimensions are said to be equivalent if nonsingular matrices T and S exist such that $A = S^{-1}BT$. Unitarily equivalence indicates that both S and T are unitary.

In this article we will study the unitary equivalence transformation of a matrix to tridiagonal form and apply this reduction to normal matrices. This transformation might seem artificial since one can always use unitary equivalences to transform matrices to bidiagonal form. However, the method seems to be useful in several instances [24–26]. An interesting historical account about this method is given in [27]. Saunders, Simon and Yip [24] discuss solving sparse unsymmetric systems of equations based on this tridiagonalization procedure. In [25], it was stated by Reichel and Ye that for particular least-squares problems this approach might be more suitable than the standard bidiagonalization procedure due to the extra created freedom. In Golub, Stoll and Wathen [26] this method was discussed for solving two systems of equations involving A and A^T simultaneously; they reconsider the tridiagonalization procedure and make the link with a block Lanczos algorithm of step size 2.

In the present article the tridiagonalization procedure is also discussed, but from a more theoretical viewpoint. Some known results are briefly reviewed and new results such as an alternative proof of the essential uniqueness of the tridiagonalization procedure are given. These results are necessary, since they will be used in the parts related to normal matrices.

The main results of this article are related to applying this tridiagonalization procedure to normal matrices. The resulting tridiagonal matrix yields interesting properties related to its super- and subdiagonal elements. It will be shown that the corresponding super- and subdiagonal elements will have the same absolute value. Even though equivalence transformations are naturally linked with singular values, we will see that for the normal case there are also tight connections with the eigenvalues when applying the reduction procedure to specific matrix classes. Flexibility in the unitary equivalence reduction will be exploited to obtain specific outcomes in case the algorithm is applied to symmetric, Hermitian, skew-Hermitian, unitary, ... matrices. Interesting properties such as, e.g. an easy way of computing the unitary-complex symmetric factorization [6, 28] of the involved normal matrix are deduced. Finally some comments on the relation with singular values and eigenvalues are presented.

The article is organized as follows. Section 2 recalls the tridiagonalization procedure for arbitrary matrices. A direct Householder method, a Lanczos variant and theorems related to the essential uniqueness are given. The method is refined for normal matrices in Section 3. It is proved that the resulting matrix inherits a strong relation between super- and subdiagonal elements. Reductions to specific matrix types and their relations with scalar product spaces are explored. In Section 4 we will deduce the tridiagonalization procedure based on "cyclical" Krylov subspaces. First cyclical Krylov subspaces are defined, followed by an analysis stating that a unitary basis for these subspaces can be used for transforming a matrix to a unitary equivalent tridiagonal form. Vice versa it is shown that any unitary equivalence transformation to tridiagonal form is coming from cyclical Krylov subspaces. In Section 5 some extra properties of the tridiagonalization procedure are presented. Section 6 discusses how to compute the singular values of a normal matrix using techniques discussed in the article. The final section contains some conclusions.

[2]This means that the bandwidth increases as one travels along the diagonal from the upper-left to the lower-right corner.

2. Preliminary results: Unitary equivalence with tridiagonal form

In this section we will analyze a unitary equivalence transformation of an arbitrary matrix into a tridiagonal matrix. This method was firstly discussed in [24] for solving systems of unsymmetric equations. The results in Sections 2.2 are fully contained in [24]. We refer the interested reader to this article for a detailed analysis and stable implementation of this method.

To be complete we include also the non-iterative variant based on Householder transformations for tridiagonalizing a matrix in Section 2.1. Most of the results are quite obvious but some extensions to the literature such as the essential uniqueness Theorem 4 are provided. This section contains, however, all necessary ingredients and preliminary results for understanding the following sections in which we will focus on the normal case. E.g., the formulas related to the Lanczos variant, the tridiagonalization procedure as well as the essential uniqueness theorem are essential in the proof of the main theorem of this article provided in Section 3.

2.1. Householder equivalence tridiagonalization

The existence of two unitary matrices U and V for reducing an arbitrary matrix to tridiagonal form is almost trivial. The algorithm involves a small adaptation of the 'well-known' standard symmetric tridiagonalization procedure [12, 13]. Instead of a similarity transformation we perform now two different unitary transformations on each side of the matrix.

We consider here the Householder tridiagonalization procedure. Assume a matrix $A \in \mathbb{C}^{n\times n}$ is given, U_k and V_k denote Householder transformation matrices of the form:

$$U_k = I - \alpha \mathbf{v}\mathbf{v}^H, \qquad V_k = I - \beta \mathbf{w}\mathbf{w}^H, \tag{1}$$

where $\alpha, \mathbf{v}, \beta$ and $\mathbf{w}$ are constructed, given an $\mathbf{x}$ and a $\mathbf{y}$ such that $U_k^H \mathbf{x} = \omega ||\mathbf{x}|| \mathbf{e}_1$, and $V_k^H \mathbf{y} = \sigma ||\mathbf{y}|| \mathbf{e}_1$. The vector $\mathbf{e}_1$ is the first standard basis vector of length equal to the length of $\mathbf{x}$, respectively, $\mathbf{y}$. The complex numbers σ and ω lie on the unit circle (i.e. $|\omega| = |\sigma| = 1$).

The following simple algorithm[3] transforms an arbitrary matrix to tridiagonal form.

Algorithm 1 (Householder equivalence tridiagonalization).
Input: Matrix A.
Output: Unitary matrices U and V and a tridiagonal T such that: $U^H A V = T$.

Set $U = I, V = I$
For k=1:n-2
  Based on $\mathbf{x} = A(k+1:n, k)$, compute the Householder reflector $U_k = I - \alpha \mathbf{v}\mathbf{v}^H$
  Set $A(k+1:n, k:n) = U_k^H A(k+1:n, k:n)$ and $U = UU_k$
  Based on $\mathbf{y} = A(k, k+1:n)^H$, compute the Householder reflector $V_k = I - \beta \mathbf{w}\mathbf{w}^H$
  Set $A(k:n, k+1:n) = A(k:n, k+1:n)V_k$ and $V = VV_k$
end

Remark 1. In the Householder equivalence tridiagonalization procedure (Algorithm 1) the resulting matrices U and V satisfy $U\mathbf{e}_1 = \mathbf{e}_1 = V\mathbf{e}_1$. This is not a constraint. Any initial unitary transformation can be applied before starting the tridiagonalization procedure. This means that, for instance one tridiagonalizes the matrix $U_0^H A V_0$ instead of A, where U_0 and V_0 are freely chosen unitary matrices. We have $U = U_0 U_1 \ldots U_{n-2}$ and $V = V_0 V_1 \ldots V_{n-2}$. As a result the equation $U\mathbf{e}_1 = \mathbf{e}_1 = V\mathbf{e}_1$ will not be true in general anymore.

2.2. Lanczos variant

Assume the following relation holds: $U^H A V = T$, for an arbitrary matrix A, T tridiagonal and both U and V unitary. Assume T has diagonal elements α_i $(i = 1, \ldots, n)$, subdiagonal elements β_i $(i = 1, \ldots, n-1)$ and superdiagonal elements γ_i $(i = 1, \ldots, n-1)$. Denote the columns of U and V as $\mathbf{u}_k$ and $\mathbf{v}_k$, for $k = 1, \ldots, n$. Based on

$$AV = UT \quad \text{and} \quad A^H U = VT^H$$

[3] MATLAB-like notation is used

we obtain the following relations:

$$A\mathbf{v}_k = \gamma_{k-1}\mathbf{u}_{k-1} + \alpha_k\mathbf{u}_k + \beta_k\mathbf{u}_{k+1} \tag{2}$$

$$A^H\mathbf{u}_k = \overline{\beta}_{k-1}\mathbf{v}_{k-1} + \overline{\alpha}_k\mathbf{v}_k + \overline{\gamma}_k\mathbf{v}_{k+1}, \tag{3}$$

for $k = 2, \ldots, n-1$ (for $k = 1$ and $k - n$ some terms do not exist and have to be ignored in the formula). Since U and V are unitary we have the following equalities with the generalized Rayleigh quotients (see e.g., [29]): $\alpha_k = \mathbf{u}_k^H A\mathbf{v}_k = \overline{\mathbf{v}_k^H A^H \mathbf{u}_k}$. Rewriting (2) and (3) gives us:

$$\begin{aligned} \mathbf{r}_{k+1} &= A\mathbf{v}_k - \gamma_{k-1}\mathbf{u}_{k-1} - \alpha_k\mathbf{u}_k, \\ \mathbf{s}_{k+1} &= A^H\mathbf{u}_k - \overline{\beta}_{k-1}\mathbf{v}_{k-1} - \overline{\alpha}_k\mathbf{v}_k. \end{aligned}$$

Hence[4] $\beta_k = \omega_k\|\mathbf{r}_{k+1}\|_2$, $\mathbf{u}_{k+1} = \mathbf{r}_{k+1}/\beta_k$ and $\gamma_k = \sigma_k\|\mathbf{s}_{k+1}\|_2$, $\mathbf{v}_{k+1} = \mathbf{s}_{k+1}/\gamma_k$, where both ω_k and σ_k are complex variables lying on the unit circle, i.e. $|\omega_k| = |\sigma_k| = 1$.

This leads to the following Lanczos-like algorithm:

Algorithm 2 (Lanczos-like unitary equivalence tridiagonalization).

Set $\mathbf{u}_0 = \mathbf{v}_0 = 0$ and $\gamma_0 = \beta_0 = 0$.
Initialize $\mathbf{u}_1$ and $\mathbf{v}_1$. (E.g., $\mathbf{u}_1 = \mathbf{e}_1 = \mathbf{v}_1$.)

for $k = 1 : n-1$
  $\alpha_k = \mathbf{u}_k^H A\mathbf{v}_k$
  $\mathbf{r} = A\mathbf{v}_k - \gamma_{k-1}\mathbf{u}_{k-1} - \alpha_k\mathbf{u}_k$
  $\mathbf{s} = A^H\mathbf{u}_k - \overline{\beta}_{k-1}\mathbf{v}_{k-1} - \overline{\alpha}_k\mathbf{v}_k$
  $\beta_k = \omega\|\mathbf{r}\|_2, \quad \gamma_k = \sigma\|\mathbf{s}\|_2$
  (ω and σ are free, satisfying $|\omega| = |\sigma| = 1$)
  $\mathbf{u}_{k+1} = \mathbf{r}/\beta_k, \quad \mathbf{v}_{k+1} = \mathbf{s}/\gamma_k$
end

This Lanczos-like tridiagonal procedure is not yet tuned for acting on normal matrices, see Section 3. Concerning details on how to implement this method using restarts and re-orthogonalization we refer to [15, 30]. Moreover an effective implementation for solving least-squares problems by this technique is discussed in [25], we refer the reader to this article for a detailed analysis of this method.

2.3. Essential uniqueness

The vectors $U\mathbf{e}_1$ and $V\mathbf{e}_1$ uniquely determine the transformation. The following theorem can be seen as an extension of the well-known implicit Q-theorem [13].

Definition 2. Two matrices $A = (a_{ij})$ and $B = (b_{ij})$ are said to be essentially identical if there exist two unitary diagonal matrices D and $\hat{D}$ such that $A = \hat{D}BD$. This means that $|a_{ij}| = |b_{ij}|$ for all i, j.

Definition 3. A tridiagonal matrix T is said to be irreducible if and only if all sub- and superdiagonal elements are different from zero.

Theorem 4. *Assume the relations $T = U^H AV$ and $S = \hat{U}^H A\hat{V}$ hold, with T and S both irreducible tridiagonal and the matrices $U, \hat{U}, V$ and $\hat{V}$ unitary. Furthermore, assume $U\mathbf{e}_1 = \hat{\omega}\hat{U}\mathbf{e}_1$ and $V\mathbf{e}_1 = \omega\hat{V}\mathbf{e}_1$, with $|\omega| = |\hat{\omega}| = 1$, then we have that the resulting tridiagonal matrices T and S are essentially identical.*

[4] The Euclidian norm is denoted by $\|\cdot\|_2$.

Proof. The proof proceeds similarly to the proof of the implicit Q-theorem in [13]. Define two new unitary matrices $W = V^H \hat{V}$ and $\hat{W} = U^H \hat{U}$. The following two equations hold:

$$TW = \hat{W}S \qquad \text{and} \qquad T^H \hat{W} = WS^H.$$

Define $\mathbf{w}_i$ and $\hat{\mathbf{w}}_i$ as the columns of W and $\hat{W}$. Writing down the equalities for the ith column we get for $i = 2, \ldots, n-1$ ($S = (s_{i,j})$):

$$\begin{aligned} T\mathbf{w}_i &= \hat{\mathbf{w}}_{i-1} s_{i-1,i} + \hat{\mathbf{w}}_i s_{i,i} + \hat{\mathbf{w}}_{i+1} s_{i+1,i}, \\ T^H \hat{\mathbf{w}}_i &= \mathbf{w}_{i-1} \overline{s}_{i,i-1} + \mathbf{w}_i \overline{s}_{i,i} + \mathbf{w}_{i+1} \overline{s}_{i,i+1}, \end{aligned}$$

which can be rewritten as

$$\hat{\mathbf{w}}_{i+1} s_{i+1,i} = T\mathbf{w}_i - \hat{\mathbf{w}}_{i-1} s_{i-1,i} - \hat{\mathbf{w}}_i s_{i,i}, \tag{4}$$

$$\mathbf{w}_{i+1} \overline{s}_{i,i+1} = T^H \hat{\mathbf{w}}_i - \mathbf{w}_{i-1} \overline{s}_{i,i-1} - \mathbf{w}_i \overline{s}_{i,i}. \tag{5}$$

In case $i = 1$ or $i = n$ some terms do not exist in Equations 4 and 5 and have to be ignored in the formula. The initial assumptions impose that $W\mathbf{e}_1 = \omega \mathbf{e}_1$ and $\hat{W}\mathbf{e}_1 = \hat{\omega}\mathbf{e}_1$. Based on the recurrence relations (4) and (5) and the fact that T is tridiagonal we get that both W and $\hat{W}$ are upper triangular. By construction both W and $\hat{W}$ are unitary. Based on the equalities $W^H W = I$ and $\hat{W}^H \hat{W} = I$ and the upper triangularity of W and $\hat{W}$ we get that both W and $\hat{W}$ are unitary diagonal. This implies $VD = \hat{V}$ and $U\hat{D} = \hat{U}$, with $W = D$ and $\hat{W} = \hat{D}$. Denote the diagonal elements of D with ω_i and the diagonal elements of $\hat{D}$ with $\hat{\omega}_i$.

Essential uniqueness of S and T follows easily (for any $1 \leq k, l \leq n$):

$$t_{k,l} = \mathbf{e}_k \hat{U}^H A \hat{V} \mathbf{e}_l = \overline{\hat{\omega}}_k \omega_l (\mathbf{e}_k U^H A V \mathbf{e}_l) = \overline{\hat{\omega}}_k \omega_l s_{k,l}.$$

Hence, $|t_{k,l}| = |s_{k,l}|$. □

Let us now consider the case in which irreducibility of S and T is not guaranteed.

Theorem 5. *Suppose $T = U^H AV$ and $S = \hat{U}^H A \hat{V}$ are both tridiagonal and the matrices $U, \hat{U}, V$ and $\hat{V}$ are unitary. Denote by K the smallest integer such that $s_{K+1,K} = 0$ and by L the smallest integer such that $s_{L,L+1} = 0$.[5] Assume $U\mathbf{e}_1 = \hat{\omega}\hat{U}\mathbf{e}_1$ and $V\mathbf{e}_1 = \omega\hat{V}\mathbf{e}_1$, with $|\omega| = |\hat{\omega}| = 1$. We have the following possibilities:*

- *$K < L$. The first K columns of U and $\hat{U}$ and the first $K+1$ columns of V and $\hat{V}$ are essentially unique. We have (for $1 \leq k \leq K$ and $1 \leq l \leq K+1$): $|t_{k,l}| = |s_{k,l}|$.*
- *$L < K$. The first $L+1$ columns of U and $\hat{U}$ and the first L columns of V and $\hat{V}$ are essentially unique. We have (for $1 \leq k \leq L+1$ and $1 \leq l \leq L$): $|t_{k,l}| = |s_{k,l}|$.*
- *$K = L$. The first K columns of U and $\hat{U}$ and the first L columns of V and $\hat{V}$ are essentially unique. We have (for $1 \leq k \leq K$ and $1 \leq l \leq L$): $|t_{k,l}| = |s_{k,l}|$.*

Proof. The proof is similar to the one of Theorem 4. We use the same notation as in the proof of Theorem 4. We will only outline the first case: $K < L$. Reconsidering Equations 4 and 5, we can only exploit Equation 4 for $2 \leq i \leq K-1$ and Equation 5 for $2 \leq i \leq K$. Equation 5 can be used for one more value of i. Hence the first K columns of $\hat{W}$ are upper triangular and the first $K+1$ columns of W are upper triangular. Therefore, the upper left $(K+1) \times (K+1)$ block of W and the upper left $K \times K$ block of $\hat{W}$ are unitary diagonal. This proves the theorem. □

Example 6. Let us illustrate which parts of the matrices T and S in Theorem 5 are essentially unique for different values of K and L. We assume T and S of dimensions 5×5. The upper left block separated from the remainder of the matrix is essential unique. This means that $|t_{ij}| = |s_{ij}|$ for elements taken out of that part of the matrix, also the zeros appear in both matrices.

[5] In case no such K exist we silently assume $K = n$. The same holds for L. We define K and L based on the matrix S, one can as well define them based on T, this does not make any difference. This choice is for convenience, w.r.t. the proof of the theorem.

$K < L$ and $K = 3$	$K > L$ and $L = 3$	$K = L = 3$
$\left[\begin{array}{cccc\|c} \times & \times & & & \\ \times & \times & \times & & \\ & \times & \times & \times & \\ \cline{1-4} & & 0 & \times & \times \\ & & & \times & \times \end{array}\right]$	$\left[\begin{array}{ccc\|cc} \times & \times & & & \\ \times & \times & \times & & \\ & \times & \times & 0 & \\ & & \times & \times & \times \\ \cline{1-3} & & & \times & \times \end{array}\right]$	$\left[\begin{array}{ccc\|cc} \times & \times & & & \\ \times & \times & \times & & \\ & \times & \times & 0 & \\ \cline{1-3} & & 0 & \times & \times \\ & & & \times & \times \end{array}\right]$

The reader can verify that this is a generalization of the implicit Q-theorem in case of Hermitian matrices [13]. In Subsection 4.2 we will provide a shorter and more appealing proof based on Krylov matrices.

Remark 7. Theorem 4 and 5 indicate that U and V can be scaled by different unitary diagonal matrices. This affects of course the resulting tridiagonal matrix T. When considering the Householder tridiagonalization procedure this flexibility can also be discovered in the construction of each Householder reflector. The reflectors can be chosen such that any ω or σ in the relations following Equation 1 can be obtained. In normal circumstances a choice is made such as to obtain the most accurate result [13, 15]. One can also choose to have $\sigma = \omega = 1$, such that one projects to a real positive number, this choice is the natural choice in the proposed Lanczos procedure.

In the remainder of the article, we will assume the most stable operation is performed. Hence we do not know whether the sub- or superdiagonals are real or not.

Everything presented in this section is directly applicable to normal matrices. Hence, we will not come back to the essential uniqueness.

3. The normal case

In the general case, the above procedure produces a tridiagonal matrix used for instance for solving sparse unsymmetric systems in [24–26]. For normal matrices, however, we will prove that $|\gamma_k| = |\beta_k|$, for the sub- and superdiagonal elements β_k and γ_k of the resulting tridiagonal matrix. We will first restrict ourselves to the irreducible case. Furthermore we will show that there is some flexibility in the reduction procedure, which can be exploited to reduce normal matrices to other specific matrix classes.

3.1. Basic theorem

The following proof is quite long and technical. Nevertheless, it provides interesting relations between the unitary transformations U and V and polynomials in the matrix A and A^H.

Theorem 8. *Suppose the matrix $A \in \mathbb{C}^{n\times n}$ is normal. Let U and V be any two unitary matrices with $U\mathbf{e}_1 = \omega V\mathbf{e}_1$ ($|\omega| = 1$) such that $U^HAV = T$, with T having subdiagonal elements β_i, superdiagonal elements γ_i and diagonal elements α_i. When all subdiagonal and superdiagonal elements are different from zero, we have $|\beta_i| = |\gamma_i|, \forall i = 1,\ldots,n-1$.*

Proof. We will prove the statement by finite induction on k ($1 \leq k \leq n-2$). We denote the columns of U and V by $[\mathbf{u}_1,\ldots,\mathbf{u}_n]$ and $[\mathbf{v}_1,\ldots,\mathbf{v}_n]$ and introduce the following notation $\beta_{1:i} = \beta_1\beta_2\ldots\beta_i$, and $\gamma_{1:i} = \gamma_1\gamma_2\ldots\gamma_i$.

In every induction step k three important items need to be proved.

(i) Initially we prove $|\gamma_k| = |\beta_k|$.
(ii) Secondly, based on the previous item, a recurrence relation using bivariate polynomials is proven for $A^H\mathbf{u}_{k+1}$ and $A\mathbf{v}_{k+1}$. More precisely we will obtain that:

$$\begin{aligned} A^H\mathbf{u}_{k+1} &= \frac{1}{\beta_{1:k}}\left(A^H\frac{\beta_{1:k-1}}{\overline{\gamma}_{1:k-1}}\,\overline{p}_k(A^H,A) - \beta_{k-1}\gamma_{k-1}p_{k-1}(A,A^H) - \alpha_k p_k(A,A^H)\right)\mathbf{v}_1 \\ &= \frac{1}{\beta_{1:k}}p_{k+1}(A,A^H)\mathbf{v}_1 \end{aligned}$$

and a similar relation

$$A\mathbf{v}_{k+1} = \frac{1}{\overline{\gamma}_{1:k}}\overline{p}_{k+1}(A^H, A)\mathbf{v}_1,$$

where $p(\cdot,\cdot)$ denotes a bivariate polynomial. With $\overline{p}(\cdot,\cdot)$ the same polynomial is meant to have complex conjugate coefficients. We initialize the recurrence with $\beta_0 = \gamma_0 = 0$, $p_0 = 0$ and $p_1(x,y) = y$. Note that $\left(p_{k+1}(A, A^H)\right)^H = \overline{p}_{k+1}(A^H, A)$.

(iii) Based on the previous two items we can prove $\|A\mathbf{v}_{k+1}\|_2 = \|A^H\mathbf{u}_{k+1}\|_2$, which concludes the induction step.

We start the inductive proof by $k = 1$. Finally we prove the statement for k assuming the relations hold for all $i = 1, 2, \ldots, k-1$. Each part of the proof is divided according to the items (i),(ii) and (iii) mentioned above.

- Suppose $k = 1$.
  (i) We have $\omega\mathbf{v}_1 = \mathbf{u}_1$. The following relations hold (since A is normal and $|\omega| = 1$):

$$\begin{aligned}\|T\mathbf{e}_1\|_2 &= \|U^H AV\mathbf{e}_1\|_2\\ &= \|A\mathbf{v}_1\|_2\\ &= \|A^H\mathbf{v}_1\|_2 = \|A^H\mathbf{u}_1\|_2\\ &= \|V^H A^H U\mathbf{e}_1\|_2 = \|T^H\mathbf{e}_1\|_2.\end{aligned}$$

  Hence, we obtain

$$|\alpha_1|^2 + |\beta_1|^2 = |\alpha_1|^2 + |\gamma_1|^2,$$

  which proves that $|\beta_1| = |\gamma_1|$.

  (ii) Secondly, we will prove the recursion formula. We have already

$$A^H\mathbf{u}_1 = p_1(A, A^H)\mathbf{v}_1 \qquad \text{and} \qquad A\mathbf{v}_1 = \overline{p}_1(A^H, A)\mathbf{v}_1.$$

  Based on (2), (3) we get

$$\beta_1\mathbf{u}_2 = A\mathbf{v}_1 - \alpha_1\mathbf{u}_1 \qquad \text{and} \qquad \overline{\gamma}_1\mathbf{v}_2 = A^H\mathbf{u}_1 - \overline{\alpha}_1\mathbf{v}_1.$$

  Multiplying the first equation by A^H and the second by A we get

$$\begin{aligned}A^H\mathbf{u}_2 &= \frac{1}{\beta_1}\left(A^H A\mathbf{v}_1 - \alpha_1 A^H\mathbf{u}_1\right)\\ &= \frac{1}{\beta_1}\left(A^H\overline{p}_1(A^H, A) - \alpha_1 p_1(A, A^H)\right)\mathbf{v}_1 = \frac{1}{\beta_1}p_2(A, A^H)\mathbf{v}_1, \qquad (6)\\ A\mathbf{v}_2 &= \frac{1}{\overline{\gamma}_1}\left(AA^H\mathbf{u}_1 - \overline{\alpha}_1 A\mathbf{v}_1\right)\\ &= \frac{1}{\overline{\gamma}_1}\left(Ap_1(A, A^H) - \overline{\alpha}_1\overline{p}_1(A^H, A)\right)\mathbf{v}_1 = \frac{1}{\overline{\gamma}_1}\overline{p}_2(A^H, A)\mathbf{v}_1. \qquad (7)\end{aligned}$$

  Note that $(p_2(A, A^H))^H = \overline{p}_2(A^H, A)$.

  (iii) Finally we prove that $\|A\mathbf{v}_2\|_2 = \|A^H\mathbf{u}_2\|_2$. Plugging the Relations (6) and (7) into $\|A\mathbf{v}_2\|_2$, using the fact that the polynomials p_2 and $\overline{p}_2$ commute (since A is normal) and using the equality $|\beta_1| = |\gamma_1|$ gives us:

$$\begin{aligned}\|A\mathbf{v}_2\|_2 &= \|\overline{p}_2(A^H, A)\mathbf{v}_1\|_2/|\gamma_1|\\ &= \left(\mathbf{v}_1^H p_2(A, A^H)\overline{p}_2(A^H, A)\mathbf{v}_1\right)/|\gamma_1|\\ &= \left(\mathbf{v}_1^H\overline{p}_2(A^H, A)p_2(A, A^H)\mathbf{v}_1\right)/|\beta_1| = \|A^H\mathbf{u}_2\|_2.\end{aligned}$$

  This proves the initial step for $k = 1$.

- Let us assume by induction now that the statements hold for all $i = 1, 2, \ldots, k-1$ and prove the case k.
  (i) Based on induction we have the relation $\|A\mathbf{v}_k\|_2 = \|A^H\mathbf{u}_k\|_2$. Since $|\beta_{k-1}| = |\gamma_{k-1}|$ one obtains the equality $|\beta_k| = |\gamma_k|$.
  (ii) The most difficult and technical part is proving the recurrence relation. Assume we have ($\forall i = 1, \ldots, k$):

$$A^H\mathbf{u}_i = \frac{1}{\beta_{1:i-1}} p_i(A, A^H)\mathbf{v}_1 \qquad \text{and} \qquad A\mathbf{v}_i = \frac{1}{\overline{\gamma}_{1:i-1}} \overline{p}_i(A^H, A)\mathbf{v}_1.$$

Based on (2) and (3), we obtain the following relations

$$\begin{aligned} A^H\mathbf{u}_{k+1} &= \frac{1}{\beta_k}\left(A^HA\mathbf{v}_k - \gamma_{k-1}A^H\mathbf{u}_{k-1} - \alpha_k A^H\mathbf{u}_k\right) \\ &= \frac{1}{\beta_{1:k}}\left(A^H\frac{\beta_{1:k-1}}{\overline{\gamma}_{1:k-1}}\overline{p}_k(A^H, A) - \gamma_{k-1}\beta_{k-1}p_{k-1}(A, A^H) - \alpha_k p_k(A, A^H)\right)\mathbf{v}_1 \\ &= \frac{1}{\beta_{1:k}} p_{k+1}(A, A^H)\mathbf{v}_1, \end{aligned} \tag{8}$$

and

$$\begin{aligned} A\mathbf{v}_{k+1} &= \frac{1}{\overline{\gamma}_k}\left(AA^H\mathbf{u}_k - \overline{\beta}_{k-1}A\mathbf{v}_{k-1} - \overline{\alpha}_k A\mathbf{v}_k\right) \\ &= \frac{1}{\overline{\gamma}_{1:k}}\left(A\frac{\overline{\gamma}_{1:k-1}}{\beta_{1:k-1}}p_k(A, A^H) - \overline{\gamma}_{k-1}\overline{\beta}_{k-1}\overline{p}_{k-1}(A^H, A) - \overline{\alpha}_k\overline{p}_k(A^H, A)\right)\mathbf{v}_1 \\ &= \frac{1}{\overline{\gamma}_{1:k}}\overline{p}_{k+1}(A^H, A)\mathbf{v}_1. \end{aligned} \tag{9}$$

The last equality is clear for the last 2 terms, for the first term we need $|\gamma_k| = |\beta_k|$ and therefore, $\beta_k/\overline{\gamma}_k = \gamma_k/\overline{\beta}_k$. Note, that again we have $\left(p_{k+1}(A, A^H)\right)^H = \overline{p}_{k+1}(A^H, A)$.
  (iii) Finally we prove that $\|A\mathbf{v}_{k+1}\|_2 = \|A^H\mathbf{u}_{k+1}\|_2$. The Relations (8) and (9) give us the following:

$$\begin{aligned} \|A\mathbf{v}_{k+1}\|_2 &= \|\overline{p}_{k+1}(A^H, A)\mathbf{v}_1\|_2/|\gamma_{1:k}| \\ &= \left(\mathbf{v}_1^H p_{k+1}(A, A^H)\overline{p}_{k+1}(A^H, A)\mathbf{v}_1\right)/|\gamma_{1:k}| \\ &= \left(\mathbf{v}_1^H \overline{p}_{k+1}(A^H, A)p_{k+1}(A, A^H)\mathbf{v}_1\right)/|\beta_{1:k}| = \|A^H\mathbf{u}_{k+1}\|_2. \end{aligned}$$

Since the above inductive procedure was finite: $k \leq n-2$, we do not yet have the equality for $|\gamma_{n-1}|$ and $|\beta_{n-1}|$. We have, however, $\|A\mathbf{v}_{n-1}\|_2 = \|A^H\mathbf{u}_{n-1}\|_2$ which gives us the desired equality.

This proves the theorem. □

It was not mentioned in the proof, but the polynomials $p_k(A, A^H)$ are also normal [3]. In fact we have even a stronger result. Since $A^H = q(A)$, with $q(\cdot)$ a polynomial of degree at most $n-1$ we can modify the proof of the theorem such that no bivariate polynomials are needed.

It is also clear that the resulting tridiagonal matrices are not necessarily normal anymore, the matrix T can be normal in specific cases as shown in Section 3.2.

Let us take a closer look at the structure of the matrix during the reduction to tridiagonal form. We will focus on the Householder reduction (Subsection 2.1). The Lanczos tridiagonalization proceeds similarly (Subsection 2.2) and since all tridiagonalization procedures are essentially equivalent (see Section 2), there is no loss of generality in this assumption. We denote by $A_k = U_{0:k}^H\, A\, V_{0:k}$, which has the upper $(k+2) \times (k+2)$ block already of tridiagonal form. Note that the upper $(k+1) \times (k+1)$ block of A_k is already in the correct form and it will not be affected anymore by any of the subsequent transformations. In each step to go from A_k to A_{k+1} we will simply apply Householder transformations as described in Section 2.1. Hence, the matrices $U_{0:k} = U_0U_1\cdots U_k$ and $V_{0:k} = V_0V_1\cdots V_k$ are a product of several Householder transformation matrices U_k and V_k. We have $U = U_{0:n-2}$ and $V = V_{0:n-2}$. The initial transformations U_0 and V_0 are somehow arbitrary, only $U_0\mathbf{e}_1 = \omega V_0\mathbf{e}_1$ is required. The matrix U has columns $\mathbf{u}_k$ and V has columns $\mathbf{v}_k$. Due to the structure of the Householder transformation matrices (see Section 2.1) we have that

$$\begin{aligned} U_{0:k}\,[\mathbf{e}_1,\ldots,\mathbf{e}_{k+1}] &= U[\mathbf{e}_1,\ldots,\mathbf{e}_{k+1}] = [\mathbf{u}_1,\ldots,\mathbf{u}_{k+1}],\\ V_{0:k}\,[\mathbf{e}_1,\ldots,\mathbf{e}_{k+1}] &= V[\mathbf{e}_1,\ldots,\mathbf{e}_{k+1}] = [\mathbf{v}_1,\ldots,\mathbf{v}_{k+1}]. \end{aligned}$$

Remark 9. In each step of the inductive proof of the theorem we do not really need the full reduced tridiagonal matrix T, the partially reduced matrix A_k having the upper left $(k+1)\times(k+1)$ block tridiagonal, is sufficient. One can check in the proof of Theorem 8 at step k that $\|A_k\mathbf{e}_{k+1}\|_2 = \|A_k^H\mathbf{e}_{k+1}\|_2$. But the equality $\|A_k\mathbf{e}_i\|_2 = \|A_k^H\mathbf{e}_i\|_2$, with $k+1 < i \le n$ does not necessarily hold. Only in specific cases equality can occur.

For a normal matrix A we always have $\|A\mathbf{e}_k\|_2 = \|A^H\mathbf{e}_k\|_2$ $(1 \le k \le n)$. But after performing the first transformation, the matrix A_1 does not satisfy $\|A_1\mathbf{e}_k\|_2 = \|A_1^H\mathbf{e}_k\|_2$ $(2 < k \le n)$ in general anymore. The equality in norm is only reestablished for a certain column $k+1$ if the column k was brought to tridiagonal form.

In the remainder no constraints will be posed on the value of both β_k and γ_k. Unfortunately the theorem holds only when $|\beta_k| = |\gamma_k|$ is different from zero. An easy counterexample consists of prepending a normal matrix by a zero column and row. The resulting matrix is still normal, but one can easily construct an equivalence transformation for which the theorem does not hold anymore.

One can, however, overcome the problem. The proof breaks down since the recursions between the vectors $\mathbf{u}_i$ and $\mathbf{v}_i$ do not hold anymore. Hence, we cannot prove by induction that $\|A\mathbf{v}_{k+1}\|_2 = \|A^H\mathbf{u}_{k+1}\|_2$ anymore, which is essential for proving the equivalence $|\beta_{k+1}| = |\gamma_{k+1}|$. When we are able to reestablish this equality in norms, we can proceed. Let us consider this in more detail.

Assume $|\beta_k| = |\gamma_k| = 0$. The following matrix is obtained after having performed unitary transforms $U_{0:k-1}$ and $V_{0:k-1}$:

$$U_{0:k-1}^H A V_{0:k-1} = A_{k-1} = \begin{bmatrix} \ddots & \ddots & & & & & & \\ \ddots & \alpha_{k-2} & \gamma_{k-2} & & & & & \\ & \beta_{k-2} & \alpha_{k-1} & \gamma_{k-1} & & & & \\ & & \beta_{k-1} & \alpha_k & & & & \\ & & & & \times & \times & \times & \cdots \\ & & & & \times & \times & \times & \cdots \\ & & & & \times & \times & \times & \cdots \\ & & & & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The next Householder reflectors U_k and V_k were initially intended to create zeros in column k and row k. Since there are already zeros we can choose them freely, acting only on rows and columns $k+1$ up to n. Considering step k in the inductive proof of Theorem 8, we see that (i) holds, (ii) cannot be completed and (iii) is undetermined. If we can construct unitary matrices U_k and V_k such that (iii) is satisfied, the proof can be continued. One can think of this as a sort of restart. When $\|A\mathbf{v}_{k+1}\|_2 = \|A^H\mathbf{u}_{k+1}\|_2$ we can continue the inductive procedure.

Since $V_{0:k} = V_{0:k-1}V_k$ and $U_{0:k} = U_{0:k-1}U_k$ and we cannot change $V_{0:k-1}$ and $U_{0:k-1}$ anymore, the vectors $\mathbf{v}_{k+1}$ and $\mathbf{u}_{k+1}$ are fully determined by the $(k+1)$th column of respectively V_k and U_k. By construction we know that $V_k\mathbf{e}_{k+1}$ and $U_k\mathbf{e}_{k+1}$ have the first k elements equal to zero. Let us therefore partition these columns as follows: $V_k\mathbf{e}_{k+1} = [0, \hat{\mathbf{v}}^T]^T$ and $U_k\mathbf{e}_{k+1} = [0, \hat{\mathbf{u}}^T]^T$, where both $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ are column vectors of length $n-k$.

Let us partition the matrices $V_{0:k-1}$ and $U_{0:k-1}$ accordingly. The superscript (l) stands for the left k columns, the superscript (r) refers to the right $n-k$ columns:

$$V_{0:k-1} = \left[V_{0:k-1}^{(l)}, V_{0:k-1}^{(r)}\right] \quad \text{and} \quad U_{0:k-1} = \left[U_{0:k-1}^{(l)}, U_{0:k-1}^{(r)}\right]. \tag{10}$$

Let us take an arbitrary $\hat{\mathbf{v}}$ different from zero, and let us see that we can construct a vector $\hat{\mathbf{u}}$ such that $\|A\mathbf{v}_{k+1}\|_2 = \|A^H\mathbf{u}_{k+1}\|_2$ holds.

Based on the partitioning for $V_{0:k-1}$ and $U_{0:k-1}$ (Equations (10)) we obtain the following equivalent relations (recall that A_k is block diagonal):

$$\begin{aligned}
\|A\mathbf{v}_{k+1}\|_2 &= \|A^H\mathbf{u}_{k+1}\|_2 \\
\|U_{0:k-1}^H A V_{0:k-1} V_k \mathbf{e}_{k+1}\|_2 &= \|V_{0:k-1}^H A^H U_{0:k-1} U_k \mathbf{e}_{k+1}\|_2 \\
\|A_k V_k \mathbf{e}_{k+1}\|_2 &= \|A_k U_k \mathbf{e}_{k+1}\|_2 \\
\left\|\left(U_{0:k-1}^{(r)}\right)^H A V_{0:k-1}^{(r)} \hat{\mathbf{v}}\right\|_2 &= \left\|\left(V_{0:k-1}^{(r)}\right)^H A^H U_{0:k-1}^{(r)} \hat{\mathbf{u}}\right\|_2 \\
\|A V_{0:k-1}^{(r)} \hat{\mathbf{v}}\|_2 &= \|A^H U_{0:k-1}^{(r)} \hat{\mathbf{u}}\|_2.
\end{aligned}$$

Since the matrix A is normal, we only need to enforce that the equality $U_{0:k-1}^{(r)}\hat{\mathbf{u}} = V_{0:k-1}^{(r)}\hat{\mathbf{v}}$ holds. Given an arbitrary $\hat{\mathbf{v}}$ we can therefore define $\hat{\mathbf{u}}$ as $\hat{\mathbf{u}} = \left(U_{0:k-1}^{(r)}\right)^H V_{0:k-1}^{(r)}\hat{\mathbf{v}}$. Based on this relation the desired equality in norms $\|A\mathbf{v}_{k+1}\|_2 = \|A^H\mathbf{u}_{k+1}\|_2$ is established.

One can continue the reduction procedure and the proof once the vectors $\hat{\mathbf{v}}$ and $\hat{\mathbf{u}}$ are embedded into two unitary transformations V_k and U_k (see e.g. [15]) both having the upper left $k \times k$ block equal to the identity matrix.

3.2. Reduction to specific matrix types

In this section some particular cases will be studied. We assume that in case the matrix T is reducible, the process is continued in such a way that equality between the sub- and superdiagonal elements still holds.

The exposition in this section draws from [31, 32] and uses results related to matrices and scalar product spaces [31, Section 2.1]. Some extra definitions are required. Let us define the bilinear form $\langle \cdot, \cdot \rangle_\Omega$ as $\langle \mathbf{x}, \mathbf{y} \rangle_\Omega = \mathbf{x}^T \Omega \mathbf{y}$, where $\cdot^T$ denotes, as before, the standard matrix transpose[6]. We assume the bilinear form to be nondegenerate, this means that Ω is nonsingular. When Ω is diagonal we will shortly refer to the bilinear form as a scalar product with weight matrix Ω. The adjoint of a matrix A with regard to $\langle \cdot, \cdot \rangle_\Omega$ is the matrix $A^\star$ such that $\langle A\mathbf{x}, \mathbf{y} \rangle_\Omega = \langle \mathbf{x}, A^\star\mathbf{y} \rangle_\Omega$, for $\mathbf{x}, \mathbf{y} \in \mathbb{F}^n$. Let $\mathbb{F}$ be either $\mathbb{C}$ or $\mathbb{R}$. A closed formula for the adjoint exists:

$$A^\star = \Omega^{-1} A^T \Omega. \tag{11}$$

Shortly, we will say adjoint with regard to the weight matrix Ω. The matrix A is said to be self-adjoint if $A^\star = A$. Based on this notation we can provide a more compact formulation of Theorem 8.

We remark that when considering normal matrices in $\mathbb{R}^{n\times n}$ we implicitly assume the transformations U and V to be real orthogonal.

Theorem 10. *Suppose the matrix $A \in \mathbb{C}^{n\times n}$ is normal. Let U and V be any unitary matrices, with $V\mathbf{e}_1 = \omega\, U\mathbf{e}_1$ ($|\omega| = 1$) such that $U^H A V = T$. Then there exists a unitary diagonal Ω such T is self-adjoint with regard to a scalar product $\langle \cdot, \cdot \rangle_\Omega$.*

Proof. The notation of Theorem 8 is used. We have that the absolute values of the sub- and superdiagonal elements are identical. This allows us to write T as the product of a complex symmetric matrix S and a diagonal matrix D: $T = SD$. When denoting $\delta_i = \beta_i/\gamma_i$ we have for instance $\overline{D} = \operatorname{diag}(1, \overline{\delta}_1, \overline{\delta}_1\overline{\delta}_2, \overline{\delta}_1\overline{\delta}_2\overline{\delta}_3, \ldots)$ leading to the desired equality. We remark that the matrix D is not unique. However, when one of its subdiagonal elements is chosen, all the remaining diagonal elements are fixed[7]. Plugging $T = SD$ into Equation 11 with $\Omega = D$ leads to the conclusion that $T^\star = T$ and hence is self-adjoint with respect to the weight matrix $\Omega = D$. The matrix Ω is unitary diagonal. □

The factorization $T = SD$ in the above proof is a complex symmetric unitary decomposition (see Section 5.2) of the matrix T (see [1, 2, 28]).

Since the unitary transformations U and V for transforming the normal matrix to tridiagonal form are not uniquely determined there is some freedom. We can exploit this freedom to obtain a stronger result.

[6]In [31] also results related to sesquilinear $\langle \mathbf{x}, \mathbf{y} \rangle_\Omega = \mathbf{x}^H \Omega \mathbf{y}$ forms are presented. For our purpose bilinear is sufficient.

[7]Note that when choosing one diagonal element freely, that its absolute value should equal 1 for the theorem to hold.

Theorem 11. *Suppose the matrix $A \in \mathbb{C}^{n\times n}$ is normal. For every given unitary diagonal matrix Ω there exist two unitary matrices U and V, with $V\mathbf{e}_1 = \omega\, U\mathbf{e}_1$, ($|\omega| = 1$) such that $U^H AV = T$, where T is self-adjoint with regard to the scalar product $\langle\cdot,\cdot\rangle_\Omega$.*

Proof. Perform a tridiagonalization procedure as provided in Theorem 8. We have $\hat{T} = \hat{U}^H A\hat{V}$. From Theorem 10 we know that $\hat{T}$ can be written as $\hat{T} = \hat{S}\hat{D}$, where $\hat{S}$ is complex symmetric and $\hat{D}$ is unitary diagonal.

Define $U = \hat{U}$, $T = \hat{T}\hat{D}^{-1}\Omega^{-1} = \hat{S}\Omega$ and $V = \hat{V}\hat{D}^{-1}\Omega$. This gives us:

$$U^H AV = \hat{U}^H A\hat{V}\hat{D}^{-1}\Omega = \hat{T}\hat{D}^{-1}\Omega = \hat{S}\Omega = T.$$

Hence, T is a tridiagonal matrix written as the product of a complex symmetric matrix $\hat{S}$ and a unitary diagonal matrix Ω. Both U and V are still unitary with $U\mathbf{e}_1 = \hat{\omega}V\mathbf{e}_1$ ($|\hat{\omega}| = 1$) and one can verify that the matrix T is self-adjoint with regard to the weight matrix Ω. □

Let us now consider a few specific matrices Ω, leading to particular unitary equivalences between A and T.

Corollary 12. *Under the conditions of Theorem 11 one can obtain T of complex symmetric form and hence self-adjoint for the standard scalar product. This means that the weight matrix is the identity.*

In fact we have for $A \in \mathbb{C}^{n\times n}$, T of complex symmetric form and for $A \in \mathbb{R}^{n\times n}$, T of symmetric form. We will refer to this reduction as the symmetric reduction. Before continuing we will shortly explain the upcoming nomenclature by a few examples. A more elaborate study and definition of these matrices can be found in [31]. In fact they are defined as being, e.g., self-adjoint or skew-adjoint, with regard to a specific weight matrix.

A matrix T is pseudo-symmetric if $T = SD$, with S symmetric and D a signature matrix. A signature matrix is a diagonal matrix having diagonal elements either 1 or −1. This matrix satisfies $T^\star = T$, with regard to the weight matrix D. A matrix T is complex pseudo-skew-symmetric if $T = SD$, where S is complex skew-symmetric and D is a signature matrix. This matrix satisfies $T^\star = -T$, with regard to the weight matrix D. A matrix T is pseudo-Hermitian if it can be written as $T = SD$, with S Hermitian and D a signature matrix. A pseudo-Hermitian matrix can also be seen as being self-adjoint with regard to a specific weight, this involves, however, the use of sesquilinear forms and a slightly modified definition of self-adjointness. We refer the reader to [31] and will not elaborate on this further in the text.

Corollary 13. *Under the conditions of Theorem 11 one can obtain T having sub- and superdiagonal elements differing only for the sign. This means that T is complex pseudo-symmetric and self-adjoint for the scalar product $\langle\cdot,\cdot\rangle_D$ in which D is a signature matrix.*

Again we have for $A \in \mathbb{C}^{n\times n}$ that T will be complex pseudo-symmetric and for $A \in \mathbb{R}^{n\times n}$ that T will be pseudo-symmetric. We will refer to this reduction as the pseudo-symmetric reduction. The sign relation between super- and subdiagonal elements can be chosen freely, for instance one can demand that they are of opposite sign. In this case the weight matrix Σ has diagonal elements $(-1)^{i+1}$, for $i = 1,\ldots,n$. This will be denoted as the skew-symmetric reduction.

Corollary 14. *Under the conditions of Theorem 11 one can obtain T having sub- and superdiagonal elements as complex conjugates (or minus the complex conjugates).*

We will refer to these reductions as the Hermitian and skew-Hermitian reductions. Similarly one can also derive a pseudo-Hermitian reduction.

The justification of the choice of names will become clear in Table 1. In the upcoming examples some of the results presented in the table will be discussed in more detail.

In Table 1 the application of a specific reduction to a specific normal matrix structure is summarized. The top row contains the possible reductions (including the weight matrix and the relation between sub- and superdiagonal elements). The first column contains the type of matrix we are performing the reduction on. The intersections depict the structure of the resulting tridiagonal matrix. In case no particular name for that special matrix structure exists a X is printed.

For simplicity we will assume in the Examples 15–17 that the resulting tridiagonal matrices are irreducible.

Table 1: Possible outcome of the reductions and the resulting structure of tridiagonal matrix.

Matrix Type	$\mathbb{F}$	**Specific Reduction Types** (Ω) Relations for γ_i and β_i: **Arb.** (Ω) $\lvert\gamma_i\rvert = \lvert\beta_i\rvert$	**Sym.** ($\Omega = I$) $\gamma_i = \beta_i, \quad \gamma_i, \beta_i \in \mathbb{R}$	**Pseu.-Sym.** ($\Omega = D$) $\gamma_i = \pm\beta_i, \quad \gamma_i, \beta_i \in \mathbb{R}$	**Skew-Sym.** ($\Omega = \Sigma$) $\gamma_i = -\beta_i, \quad \gamma_i, \beta_i \in \mathbb{R}$
Normal	$\mathbb{R}$	Pseu.-Sym.	Sym.	Pseu.-Sym.	Pseu.-Sym.
Sym.	$\mathbb{R}$	Pseu.-Sym.	Sym.	Pseu.-Sym.	Pseu.-Sym.
Skew-Sym.	$\mathbb{R}$	Pseu.-Skew-Sym	Pseu.-Skew-Sym.	Pseu.-Skew-Sym.	Skew-Sym.
Orthogonal	$\mathbb{R}$	Pseu.-Sym. Orth. Block-Diag.	Sym. Orth. Block-Diag.	Pseu.-Sym. Orth. Block-Diag.	Pseu.-Sym Orth. Block-Diag.
Normal	$\mathbb{C}$	X	Cplx.-Sym.	Cplx. Pseu.-Sym.	Cplx. Pseu.-Sym.
Herm.	$\mathbb{C}$	X	Cplx.-Sym.	Cplx. Pseu.-Sym.	Cplx. Pseu.-Sym.
Skew-Herm.	$\mathbb{C}$	X	Cplx.-Sym.	Cplx. Pseu.-Sym.	Cplx. Pseu.-Sym.
Unitary	$\mathbb{C}$	X Unit. Block-Diag.	Cplx.-Sym. Unit. Block-Diag.	Cplx. Pseu.-Sym. Unit. Block-Diag.	Cplx. Pseu.-Sym. Unit. Block-Diag.

Matrix Type	$\mathbb{F}$	**Specific Reduction Types** Relations for γ_i and β_i: **Herm.** $\gamma_i = \overline{\beta}_i, \quad \gamma_i, \beta_i \in \mathbb{C}$	**Pseu.-Herm.** $\gamma_i = \pm\overline{\beta}_i, \quad \gamma_i, \beta_i \in \mathbb{C}$	**Skew-Herm.** $\gamma_i = -\overline{\beta}_i, \quad \gamma_i, \beta_i \in \mathbb{C}$
Normal	$\mathbb{R}$	Sym.	Pseu.-Sym.	Pseu.-Sym.
Sym	$\mathbb{R}$	Sym.	Pseu.-Sym.	Pseu.-Sym.
Skew-Sym.	$\mathbb{R}$	Pseu.-Skew-Sym.	Pseu.-Skew-Sym.	Skew-Sym.
Orthogonal	$\mathbb{R}$	Sym. Orth. Block-Diag.	Pseu.-Sym Orth. Block-Diag.	Pseu.-Sym Orth. Block-Diag.
Normal	$\mathbb{C}$	X	X	X
Herm.	$\mathbb{C}$	Herm.	Pseu.-Herm.	Pseu.-Herm.
Skew-Herm.	$\mathbb{C}$	Pseu.-Skew-Herm.	Pseu.-Skew-Herm.	Skew-Herm.
Unitary	$\mathbb{C}$	Unitary Block-Diag.	Unitary Block-Diag.	Unitary Block-Diag.

Example 15. Suppose A is symmetric and the symmetric reduction $U^T AV = T$ is applied. Since the matrix T is real we clearly have that T is symmetric. This proves the relation depicted in the table. In fact we have even more. Due to the symmetry of T we get $U^T AV = T = T^T = V^T AU$. Hence we have two different reductions applied on the matrix A, both resulting in a tridiagonal matrix. Since $U\mathbf{e}_1 = \pm V\mathbf{e}_1$ by construction, we can apply Theorem 4 and we get $UD = V$, with D a signature matrix. Since T is symmetric one can easily deduce that $D = -I$ or $D = I$, depending on $U\mathbf{e}_1 = \pm V\mathbf{e}_1$. Hence $U = \pm V$ and the standard orthogonal similarity transformation of a symmetric matrix to symmetric tridiagonal form is obtained when $U\mathbf{e}_1 = V\mathbf{e}_1$.

Example 16. Suppose A is skew-symmetric and we apply the symmetric reduction: $U^T AV = T$. Table 1 states that the resulting tridiagonal will be pseudo skew-symmetric. The pseudo-structure is obvious, only the skew-symmetric structure implies the diagonal elements to be zero. Since $A = -A^T$ we obtain $U^T AV = T$ and $V^T AU = -T$. Applying the essential uniqueness theorem gives us $UD = V$. Therefore $V^T AV = TD$, with D a signature matrix. Moreover, since A is skew-symmetric, the matrix product TD is also skew-symmetric. Therefore, the diagonal elements of T will be zero.

Example 17. Assume A to be skew-Hermitian and we apply the pseudo-Hermitian reduction to the matrix. We are specifically interested in the diagonal elements of T since the table states that they are purely imaginary. Similar arguments as in the previous examples lead to $U^H AV = T$ and $V^H AU = -T^H$. Hence $UD = V$, with D unitary diagonal by Theorem 4. Therefore we have $U^H AU = T\overline{D}$, which is skew-Hermitian. This implies that $T\overline{D}$ is skew-Hermitian. We have $-T\overline{D} = DT^H$ and we know the relation between the sub- and superdiagonal elements we have that $D = \overline{D}$ is a signature matrix, this implies in turn that the diagonal elements of T need to be purely imaginary. Hence the resulting tridiagonal matrix T will be pseudo skew-Hermitian.

Example 18. Suppose the matrix A to be unitary: $AA^H = I$. In this case we obtain a unitary complex symmetric tridiagonal matrix. One can easily verify that this tridiagonal matrix cannot be irreducible (assume $n > 2$). The resulting tridiagonal matrix will be of block diagonal form, having block diagonals, which are 2×2 unitary matrices or 1×1 complex numbers lying on the unit circle. In Section 4.3 we will even show that in practice the tridiagonal matrix will normally have 2×2 blocks on the diagonal, and eventually a trailing 1×1 block in case of odd matrix size.

4. Krylov subspace approach

In the previous section the Lanczos approach was deduced based on the Householder tridiagonalization scheme. Here, we will construct two Krylov sequences and prove that an orthonormal basis for these Krylov subspaces will tridiagonalize the matrix. Based on the orthonormalization procedure of these Krylov subspaces one obtains again the Lanczos variant as described in Section 2.2. The results of Section 4.1 are contained in a more elaborate form in [24–26], we will only consider the case in which no breakdowns occur. In Section 4.2 we will discuss a new more appealing proof of the essential uniqueness Theorem 4, which was not discussed in any of the above articles. In Section 4.3 we will present some examples related to normal matrices.

4.1. Cyclical Krylov subspaces

We start first by studying arbitrary matrices, afterwards we specialize towards the normal case.

Assume we have the following cyclical Krylov sequences[8]:

$$\begin{aligned} C_k(A, \mathbf{x}, \mathbf{y}) &= \operatorname{span}\{\mathbf{x}, A\mathbf{y}, AA^H\mathbf{x}, AA^HA\mathbf{y}, (AA^H)^2\mathbf{x}, \ldots\}, \\ C_k(A^H, \mathbf{y}, \mathbf{x}) &= \operatorname{span}\{\mathbf{y}, A^H\mathbf{x}, A^HA\mathbf{y}, A^HAA^H\mathbf{x}, (AA^H)^2\mathbf{y}, \ldots\}. \end{aligned}$$

Even though not specified in the above sequence, the subscript k denotes the number of vectors that generate the kth subspace. For simplicity we assume in the following that the dimension of both cyclical Krylov subspaces $C_k(A, \mathbf{x}, \mathbf{y})$ and $C_k(A^H, \mathbf{y}, \mathbf{x})$ is equal to k. For the more general case we refer to [24–26].

[8]In [24] four different spaces are defined depending on the odd and even vectors. Their approach coincides in fact with this one. These spaces are not defined in [25, 26]. In [26] the link is made with block Lanczos of step size 2.

We call this a cyclical sequence since the vectors $\mathbf{x}$ and $\mathbf{y}$ alternate to build up two sequences. More precisely, the ith vector of the sequence $C_k(A, \mathbf{x}, \mathbf{y})$ is multiplied by A^H and forms the $(i+1)$st vector of $C_k(A^H, \mathbf{y}, \mathbf{x})$. Conversely the ith vector of $C_k(A^H, \mathbf{y}, \mathbf{x})$ is multiplied by A resulting in the $(i+1)$st vector of $C_k(A, \mathbf{x}, \mathbf{y})$.

Construct for every $1 \leq k \leq n$ an orthonormal basis say $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_k\}$ for $C_k(A, \mathbf{x}, \mathbf{y})$, similarly construct an orthonormal basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ for $C_k(A^H, \mathbf{y}, \mathbf{x})$. Using the vectors $\mathbf{u}_i$ and $\mathbf{v}_i$ as the columns of two matrices results in the matrices U_k and V_k. We remark that the notation in this section changes substantially with regard to the one in the previous sections: the matrices U_k and V_k do not denote Householder transformations or unitary matrices anymore!

The following two important relations clearly hold:

$$AC_k(A^H, \mathbf{y}, \mathbf{x}) \subset C_{k+1}(A, \mathbf{x}, \mathbf{y}) \quad \text{and} \quad A^H C_k(A, \mathbf{x}, \mathbf{y}) \subset C_{k+1}(A^H, \mathbf{x}, \mathbf{y}). \tag{12}$$

Since $\mathbf{v}_k \in C_{k+1}(A^H, \mathbf{y}, \mathbf{x}) \backslash C_k(A^H, \mathbf{y}, \mathbf{x})$ we have $A\mathbf{v}_k \perp \mathbf{u}_i$, where $1 \leq i \leq k-2$, since $\mathbf{u}_k \in C_{k+1}(A, \mathbf{x}, \mathbf{y}) \backslash C_k(A, \mathbf{x}, \mathbf{y})$ we also have $A^H \mathbf{u}_k \perp \mathbf{v}_i$, where $1 \leq i \leq k-2$. Both relations can be proved by Equation (12) and the fact that ($\langle \cdot, \cdot \rangle$ stands again for the standard inproduct) $\langle A\mathbf{v}_k, \mathbf{u}_i \rangle = \langle \mathbf{v}_k, A^H \mathbf{u}_i \rangle$ and $\langle A^H \mathbf{u}_k, \mathbf{v}_i \rangle = \langle \mathbf{u}_k, A\mathbf{v}_i \rangle$. Considering the orthogonality relations between the vectors $\mathbf{u}_i$ and $\mathbf{v}_i$ we get (for $2 \leq i \leq k$ and assuming for now all β and γ different from zero):

$$A\mathbf{v}_i \;=\; \gamma_{i-1}\mathbf{u}_{i-1} + \alpha_i \mathbf{u}_i + \beta_i \mathbf{u}_{i+1},$$

where $\beta_{i+1} = \langle \mathbf{u}_{i+1}, A\mathbf{v}_i \rangle, \alpha_i = \langle \mathbf{u}_i, A\mathbf{v}_i \rangle$ and $\gamma_{i-1} = \langle \mathbf{u}_{i-1}, A\mathbf{v}_i \rangle$. A similar equation holds for $A\mathbf{u}_i$, also the upcoming formulas and conclusions in this section can be rewritten in terms of AU_k.

Combining all these equations into a single matrix formula gives:

$$AV_k = U_k T_k + \beta_k \mathbf{u}_{k+1} \mathbf{e}_k^T, \tag{13}$$

where T_k is a $k \times k$ tridiagonal matrix having the elements α_i on the diagonal, the β's on the subdiagonal and the γ's on the superdiagonal. Running the process to completion gives us the desired tridiagonalization: $U_n^H A V_n = U^H A V = T$.

We assumed, however, all β and γ to be different from zero. Otherwise we have a breakdown and some standard tricks are needed for restarting the procedure. We refer to [24–26, 33].

At the time of writing this article, the paper [26] appeared. Instead of considering cyclical Krylov subspaces they make the link with the following product block Krylov subspace[9]:

$$\mathcal{K}_k\left(\begin{bmatrix} 0 & A \\ A^H & 0 \end{bmatrix}, \begin{bmatrix} \mathbf{x} & 0 \\ 0 & \mathbf{y} \end{bmatrix}\right).$$

For details on product and block Krylov methods we refer to [15, 34].

4.2. Cyclical Krylov matrices

We have already shown that one can obtain the Lanczos process from the unitary tridiagonalization scheme (based, e.g., on Householder transformations) in Subsection 2.1. Furthermore we also stated in the previous subsection that the same process is obtained starting from initial cyclical Krylov subspaces. In this subsection we will prove that the unitary matrices involved in a unitary equivalence to tridiagonal form are always coming from specific cyclical subspaces. (The treatment is similar to the one in [15].) For simplicity we assume the resulting tridiagonal matrices to have both sub- and superdiagonals different from zero.

Based on cyclical Krylov subspaces, we can define cyclical Krylov matrices:

$$\begin{aligned} C_k(A, \mathbf{x}, \mathbf{y}) &= \left[\mathbf{x}, A\mathbf{y}, AA^H\mathbf{x}, AA^H A\mathbf{y}, (AA^H)^2\mathbf{x}, \ldots\right], \\ C_k(A^H, \mathbf{y}, \mathbf{x}) &= \left[\mathbf{y}, A^H\mathbf{x}, A^H A\mathbf{y}, A^H AA^H\mathbf{x}, (AA^H)^2\mathbf{y}, \ldots\right]. \end{aligned}$$

Lemma 19. *Assume $AV = U\hat{A}$ and $A^H U = V\hat{A}^H$ hold, then we have the following equalities:*

$$UC_k(\hat{A}, \mathbf{x}, \mathbf{y}) = C_k(A, U\mathbf{x}, \mathbf{y}), \qquad VC_k(\hat{A}^H, \mathbf{y}, \mathbf{x}) = C_k(A, V\mathbf{y}, \mathbf{x}).$$

[9]With $\mathcal{K}_k(A, \mathbf{x}) = \text{span}\{\mathbf{x}, A\mathbf{x}, A^2\mathbf{x}, \ldots\}$ the standard Krylov subspace is meant.

The proof involves straightforward computations. We remark that it is not necessary that U and V are unitary.

The following theorem states that the unitary matrices used in the equivalence transformation to tridiagonal form, make up an orthonormal basis for a certain cyclical Krylov subspace.

Theorem 20. *Suppose $U^H AV = T$, with U, V unitary and T tridiagonal having all sub- and superdiagonal elements different from zero. We have for every k: the columns of U_k form an orthonormal basis for $C_k(A, \mathbf{u}_1, \mathbf{v}_1)$ and the columns of V_k form an orthonormal basis for $C_k(A^H, \mathbf{v}_1, \mathbf{u}_1)$.*

Proof. We have that $C_k(T, \mathbf{e}_1, \mathbf{e}_1) = R$ and $C_k(T^H, \mathbf{e}_1, \mathbf{e}_1) = \hat{R}$, with both $\hat{R}$ and R nonsingular upper triangular. Based on Lemma 19, we obtain the following two QR-factorizations for every k:

$$UR = UC_k(T, \mathbf{e}_1, \mathbf{e}_1) = C_k(A, \mathbf{u}_1, \mathbf{v}_1), \quad \text{and} \quad V\hat{R} = VC_k(T^H, \mathbf{e}_1, \mathbf{e}_1) = C_k(A^H, \mathbf{v}_1, \mathbf{u}_1).$$

This concludes the proof. □

Interesting is that the relations above also lead to an alternative proof of the essential uniqueness Theorem 4. Assume the conditions as provided in Theorem 4 hold, i.e., $\mathbf{u}_1 = \hat{\omega}\hat{\mathbf{u}}_1$ and $\mathbf{v}_1 = \omega\hat{\mathbf{v}}_1$. Theorem 20 provides with us the following equalities ($R_U, \hat{R}_U, R_V$ and $\hat{R}_V$ are nonsingular upper triangular):

$$UR_U = C_k(A, \mathbf{u}_1, \mathbf{v}_1) = C_k(A, \overline{\hat{\omega}}\hat{\mathbf{u}}_1, \overline{\omega}\hat{\mathbf{v}}_1) = \hat{U}\hat{R}_U \quad \text{and} \quad VR_V = C_k(A, \mathbf{v}_1, \mathbf{u}_1) = C_k(A, \overline{\omega}\hat{\mathbf{v}}_1, \overline{\hat{\omega}}\hat{\mathbf{u}}_1) = \hat{U}\hat{R}_U.$$

Based on the uniqueness of the QR-factorization we know that $U\hat{D} = \hat{U}$ and $VD = \hat{V}$ for two unitary diagonal matrices $\hat{D}$ and D.

We will not go into the details but in case one of the sub- and/or superdiagonal elements is zero a similar analysis applies and results identical to the ones of Theorem 5 are obtained.

The following theorem summarizes these results.

Theorem 21. *For U and V unitary, we have that $U^H AV = T$ is tridiagonal if and only if the columns of U and V define an orthonormal basis for a specific cyclical Krylov subspace.*

The proof consists of a combination of previous results.

4.3. The normal case

We are familiar now with the generic case. Let us see what changes in the normal matrix setting. Let us consider as an example the Hermitian, skew-Hermitian and unitary case (see also [24]).

Example 22. Consider the matrix A to be Hermitian, i.e. $A = A^H$. In this case the procedure above simplifies. One obtains the following two cyclical Krylov sequences:

$$C_k(A^H, \mathbf{x}, \mathbf{x}) = C_k(A, \mathbf{x}, \mathbf{x}) \;=\; \text{span}\{\mathbf{x}, A\mathbf{x}, A^2\mathbf{x}, A^3\mathbf{x}, A^4\mathbf{x}, \ldots, A^k\mathbf{x}\}.$$

We obtain $C_k(A^H, \mathbf{x}, \mathbf{x}) = C_k(A, \mathbf{x}, \mathbf{x}) = \mathcal{K}_k(A, \mathbf{x})$. The latter sequence is just the standard Krylov subspace. Hence the method simplifies and produces nothing else than the standard Lanczos tridiagonalization procedure.

Example 23. For a skew-Hermitian matrix $A = -A^H$ we obtain the following cyclical Krylov subspaces:

$$\begin{aligned} C_k(A, \mathbf{x}, \mathbf{x}) &= \text{span}\{\mathbf{x}, A\mathbf{x}, -A^2\mathbf{x}, -A^3\mathbf{x}, A^4\mathbf{x}, \ldots\}, \\ C_k(A^H, \mathbf{x}, \mathbf{x}) &= \text{span}\{\mathbf{x}, -A\mathbf{x}, -A^2\mathbf{x}, A^3\mathbf{x}, A^4\mathbf{x}, \ldots\}. \end{aligned}$$

Clearly they equal the standard Krylov subspace $\mathcal{K}_k(A, \mathbf{x})$. Hence, the approach coincides with the standard tridiagonalization approach.

Example 24. Assume A to be unitary $AA^H = A^H A = I$. We know from Example 18 that the resulting tridiagonal matrix will be a block diagonal matrix having 2×2 blocks or 1×1 blocks on the diagonal. We distinguish between two cases: $\mathbf{v}$ is an eigenvector of A or not. If $\mathbf{v}$ is an eigenvector, it is obvious that $C_2(A, \mathbf{v}, \mathbf{v}) = C_1(A, \mathbf{v}, \mathbf{v})$ and $C_2(A^H, \mathbf{v}, \mathbf{v}) = C_1(A^H, \mathbf{v}, \mathbf{v})$ and hence we have a 1×1 block on the diagonal and a restart is required.

If $\mathbf{v}$ is not an eigenvector we have:

$$C_3(A, \mathbf{v}, \mathbf{v}) = \operatorname{span}\{\mathbf{v}, A\mathbf{v}, AA^H\mathbf{v}\} = \operatorname{span}\{\mathbf{v}, A\mathbf{v}, I\mathbf{v}\} = \operatorname{span}\{\mathbf{v}, A\mathbf{v}\} = C_2(A, \mathbf{v}, \mathbf{v})$$

and similarly $C_3(A^H, \mathbf{v}, \mathbf{v}) = C_2(A^H, \mathbf{v}, \mathbf{v})$. These invariant subspaces create a 2×2 block on the diagonal. Hence also in this case a restart is required.

We can conclude that the will obtain a tridiagonal matrix having blocks of size two at most on the diagonal. Moreover, since one will almost never succeed in starting with a vector $\mathbf{v}$ which is an eigenvector, generically the resulting tridiagonal matrix will consist of 2×2 blocks, and eventually a trailing 1×1 block when the matrix is of odd size.

5. Extra properties

The unitary equivalence transformation of a normal matrix into tridiagonal form, and especially into complex symmetric tridiagonal form implies some other interesting relations. In this section we will further explore some properties related to the reduction and we will very briefly comment on a unitary complex symmetric decomposition.

5.1. Complex symmetric matrices

In this subsection we will silently assume that the matrix $U^H A V = T$, with U, V unitary and A normal, is complex symmetric, unless stated otherwise. This transformation of a normal matrix to tridiagonal complex symmetric form can also be applied on matrices closely related to the normal matrix such as its Hermitian conjugate or its inverse and will again result in a complex symmetric matrix.

Corollary 25. *Suppose $U^H A V = T$, under the conditions of Theorem 8, with T complex symmetric and A a normal matrix having distinct singular values. Then $U^H V$ will also be complex symmetric.*

Proof. The matrix T is complex symmetric, which implies the relations $U^H A V = T = T^T = V^T A^T \overline{U}$. Reshuffling the unitary matrices U and V gives us

$$\overline{V} U^H A = A^T \overline{U} V^H, \tag{14}$$

which implies that both matrix products are also complex symmetric. For simplicity we will denote this as $XA = A^T X^T$, where $X = \overline{V} U^H$. Hence, it remains to prove that X is complex symmetric.

Assume now that we have the following singular value decomposition of the matrix A: $A = W \Sigma D_1 W^H$, where Σ a diagonal containing the singular values and D_1 a unitary diagonal matrix. We know that $\Delta = \Sigma D_1$ is a diagonal matrix containing the eigenvalues, since A is normal.

Plugging this into $XA = A^T X^T$ gives:

$$\begin{aligned} X(W \Sigma D_1 W^H) &= (\overline{W} D_1 \Sigma W^T) X, \\ (XW) \Sigma (D_1 W^H) &= (\overline{W} D_1) \Sigma (W^T X) \end{aligned}$$

The previous equation provides us two different singular value decompositions of the same matrix. Since all singular values are distinct, the decomposition is essentially unique. Hence we obtain for a unitary diagonal matrix D_2 that $XW = \overline{W} D_1 D_2$. This proves that $X = X^T$ and hence $U^H V$ is a unitary complex symmetric matrix. □

Remark 26. The previous proof implies the following interesting relation, assuming that all conditions of the theorem hold. Given the eigenvalue decomposition of A: $A = W \Delta W^H$ then we have that $W^T X W$ is unitary diagonal.

A second theorem states that applying the equivalence transformation to positive powers of A always results in a complex symmetric matrix.

Corollary 27. *Suppose $U^HAV = T$, under the conditions of Theorem 8, with T complex symmetric and A a normal matrix having distinct singular values. Then U^HA^iV will also be complex symmetric for $i \in \mathbb{N}$.*

Proof. We want to prove that $\left(U^HA^iV\right)^T = U^HA^iV$. Equation (14) can be rewritten as:

$$UV^TA^T = AVU^T. \tag{15}$$

The remainder of the proof involves standard matrix reordering techniques and uses some of the proved equalities, involving also Corollary 25:

$$\begin{aligned}
\left(U^HA^iV\right)^T &= V^T(A^T)^i\,\overline{U} \\
&= U^H(UV^TA^T)(A^T)^{i-1}\,\overline{U} \\
&= U^H(AVU^T)(A^T)^{i-1}\,\overline{U} \\
&= U^H(AUV^T)(A^T)^{i-1}\,\overline{U} \\
&= U^HA(UV^TA^T)(A^T)^{i-2}\,\overline{U} \\
&= \ldots \\
&= U^HA^iVU^T\overline{U} = U^HA^iV,
\end{aligned}$$

which is the desired equality. □

Corollary 28. *Suppose $U^HAV = T$, under the conditions of Theorem 8, with T complex symmetric and A a normal matrix having distinct singular values. We have that the following matrices will be complex symmetric. In few cases non-singularity of A is assumed.*

1. *U^HV, V^HU are complex symmetric.*
2. *U^HA^iV (with $i \in \mathbb{Z}$) is complex symmetric.*
3. *V^HA^iU (with $i \in \mathbb{Z}$) is complex symmetric.*
4. *$U^H(A^H)^iV$ (with $i \in \mathbb{Z}$) is complex symmetric.*
5. *$V^H(A^H)^iU$ (with $i \in \mathbb{Z}$) is complex symmetric.*
6. *$U^Hp(A, A^H, A^{-1})V$ is complex symmetric (p a polynomial).*
7. *$V^Hp(A, A^H, A^{-1})U$ is complex symmetric (p a polynomial).*

Proof. All relations can be proved, based on three important relations:

$$U^HV = V^T\overline{U}, \qquad \overline{V}U^HA = A^T\overline{U}V^H, \quad \text{and} \quad UV^TA^T = AVU^T.$$

For the case U^HA^HV one can use the argument that there exists a polynomial $p(\cdot)$ such that $A^H = p(A)$ (see Condition 17 in [3]). □

When applying unitary transformations U and V based on A and A^H some other relations hold.

Theorem 29. *Suppose $U^HAV = T$, under the conditions of Theorem 8, with T complex symmetric and A a normal matrix having distinct singular values. The following relation holds between $A_U = U^HAU$ and $A_V = V^HA^HV$:*

$$\overline{A_U} = A_V.$$

Proof. We have

$$\begin{aligned}
A_U &= (U^HAV)V^HU = TV^HU, \\
A_V &= (V^HA^HU)UV^H = \overline{T}UV^H = \overline{T}V^T\overline{U}.
\end{aligned}$$

Taking the complex conjugate provides the result. □

Remark 30. Based on the relations from Theorem 29 one can deduce a similarity transformation for transforming the matrix A into its transpose A^T:

$$\left(UV^T\right)A^T\left(UV^T\right)^H = A.$$

In the following T is not necessarily complex symmetric anymore.

Theorem 31. *Suppose $U^HAV = T$, under the conditions of Theorem 8 and A a normal matrix having distinct singular values. The following relation holds between $A_U = U^HAU$ and $A_V = V^HA^HV$:*

$$|A_U| = A_V.$$

Remark 32. Suppose the skew-symmetric reduction was applied to a normal matrix A, i.e. that the off-diagonal elements have opposite signs. We have the following relation between A_U and A_V:

$$Y\overline{A_U}Y = A_V.$$

with Y a diagonal matrix having diagonal elements $y_{ii} = (-1)^{i+1}$.

5.2. A unitary - complex symmetric decomposition

In [1, 2, 28] the SU-factorization $A = SU$, in which S is complex symmetric and U is unitary was presented. In fact in [28] another sort of polar-decomposition [35, 36] was proposed. The standard polar-decomposition[10] for a matrix A is of the form $A = HU$, in which H is Hermitian semi-positive definite. Under some constraints the polar-decomposition is unique. The $SUPD$-decomposition which is a complex symmetric unitary decomposition with the complex symmetric matrix semi-positive definite is studied in relation with normal and conjugate normal matrices in [6, 7, 28].

Suppose A to be a normal matrix. Since A is unitary equivalent to a complex symmetric tridiagonal matrix, the matrix A admits a SU-decomposition of the following form $A = UTV^H = \left(UV^T\right)\left(\overline{V}TV^H\right) = WP$. The factor $W = UV^T$ is obviously unitary, and $P = \overline{V}TV^H$ is complex symmetric.

6. Eigenvalues and singular values

It is already clear from the previous sections that the reduction proposed in this article is closely related to an initial step for computing for instance the eigenvalues and/or the singular values. In this section we will briefly comment on possible alternative ways for computing singular values and/or eigenvalues. Based on the unitary equivalence transformation one can transform any normal matrix to a complex symmetric tridiagonal matrix T. For computing the singular values one can proceed with the tridiagonal matrix T. Singular values of a complex symmetric tridiagonal matrix T can be computed for example with methods from [37–40]. We will briefly comment on [40] with regard to our interest.

Assume C to be a complex symmetric matrix $C = C^T$, then there exists a unitary Q, such that $C = Q\Sigma Q^T$, where Σ is a diagonal matrix having diagonal elements $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n$. These are the singular values and the factorization is often named a symmetric singular value decomposition (SSVD) or the Takagi factorization of C.

The standard SVD equals $U\Sigma V$, hence it might not come as a surprise that the method proposed in [40] can be faster than the standard SVD method, in case the unitary factors Q and Q^T are desired. Moreover, the single unitary factor Q consumes less memory than the factors U and V.

Applying the unitary equivalence reduction to tridiagonal form, followed by the method proposed in [40] leads to an alternative method for computing the singular values and singular vectors of a normal matrix.

Since eigenvalues of particular subclasses such as Hermitian, skew-Hermitian and unitary can be computed efficiently also the generic class of normal matrices is of interest. Different techniques have already been proposed. Elsner and Ikramov proposed in [19] a condensed form for normal matrices based on similarity transformations, which could

[10] In [28] one used the order HU for the polar-decomposition, it is more common to use UH.

then be exploited for developing fast QR-like methods. In [20, 22, 23] some iterative procedures were presented and analyzed.

In the previous sections we showed that the unitary equivalence presented in this article sometimes reduces to a unitary similarity transformation. Hence for the cases of Hermitian and skew-Hermitian, when computing eigenvalues, this coincides with standard techniques for reducing the bandwidth and preserving the spectrum.

Based on the full singular value decomposition, one can however also compute the eigenvalues. Assume the normal matrix A has the following singular value decomposition $A = U\Sigma V^H$, based on properties of normal matrices we know that the eigenvalues are $\Delta = \Sigma D$, where $D = V^H U$. This means that based on previous results of this section, we can compute the full eigenvalue decomposition once the full singular value decomposition is known.

7. Conclusions and future research

In this article the unitary equivalence transformation of a normal matrix to tridiagonal form was discussed. Furthermore, the transformation could be chosen in such a way that the resulting tridiagonal matrix is self-adjoint with regard to a previously defined scalar product space $\langle \cdot, \cdot \rangle_\Omega$, for a unitary diagonal matrix Ω.

A Householder tridiagonalization scheme as well as an iterative method and its relation to Krylov subspaces was presented. Several possibilities for reducing the matrices were extensively explored and applied to well-known classes of normal matrices. Extra properties related to the equivalence transformation were proved. Finally a few possibilities for exploiting the new method for computing eigenvalue and singular values were briefly discussed.

Numerical experiments as well as a more detailed analysis related to the different techniques for computing the eigenvalues and singular values were not discussed, since they were beyond the scope of this article and are subject to further research. Extra effort is needed to implement the methods, analyze their stability and computational complexity, study the convergence and so on. The reduction from normal to tridiagonal form based on Householder transformations, which is fairly straightforward to implement, can, however, be downloaded from the author's home page. The Matlab files admit different kinds of reductions, such as, e.g., skew-symmetric, skew-conjugate and so forth. The software includes extra m-files which enable the interested reader to quickly try out several of the equalities and properties provided in the article and to play with different matrices.

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